

A FAMILY OF $*$ -ALGEBRAS ALLOWING WICK ORDERING: FOCK REPRESENTATIONS AND UNIVERSAL ENVELOPING C^* -ALGEBRAS

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ABSTRACT. We consider an abstract Wick ordering as a family of relations on elements a_i and define $*$ -algebras by these relations. The relations are given by a fixed operator $T: \mathfrak{h} \otimes \mathfrak{h} \rightarrow \mathfrak{h} \otimes \mathfrak{h}$, where \mathfrak{h} is one-particle space, and they naturally define both a $*$ -algebra and an inner-product space \mathcal{H}_T , $\langle \cdot, \cdot \rangle_T$. If a_i^* denotes the adjoint, i.e., $\langle a_i \varphi, \psi \rangle_T = \langle \varphi, a_i^* \psi \rangle_T$, then we identify when $\langle \cdot, \cdot \rangle_T$ is positive semidefinite (the positivity question!). In the case of deformations of the CCR-relations (the q_{ij} -CCR and the twisted CCR's), we work out the universal C^* -algebras \mathfrak{A} , and we prove that, in these cases, the Fock representations of the \mathfrak{A} 's are faithful.

1. Introduction

In recent papers [1, 2, 3, 4, 5, 6], the applications of Lie superalgebras, quantum groups, q -algebras in mathematical physics have stimulated interest in

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the $*$ -algebras defined by generators and relations and their representations by Hilbert space operators. For example, the representations of various deformations of canonical commutation relations (CCR), in particular Fock representation, were used to construct non-classical models of theoretical physics and probability, such as the free quon gas (see [7]), q -Gaussian processes (see [8]) etc.

The constructions are interesting from both physical and mathematical points of view. They give a canonical realisation of a given deformed relation like the Fock representation, or a realisation by differential operators. When the relations can be realised by bounded operators, it is useful to study the universal enveloping C^* -algebras for them and the stability of isomorphism classes of these C^* -algebras on parameters (see for example [9, 10]). The stability question [10] refers to how the C^* -isomorphism classes depend on variations in the deformation variables; in some cases there are open regions in parameter space where the C^* -isomorphism class is constant.

In the present paper we give a review of some results concerning a wide class of deformed relations of the following form

$$a_i^* a_j = \delta_{ij} 1 + \sum_{k,l=1}^d T_{ij}^{kl} a_l a_k^*, \quad i, j = 1, \dots, d, \quad (1)$$

where $T_{ij}^{kl} \in \mathbb{C}$, such that $T_{ij}^{kl} = \bar{T}_{ji}^{lk}$. These relations generate a $*$ -algebra allowing Wick ordering or **Wick algebra** (see [4, 11, 12, 13]). The $*$ -algebra \mathfrak{A}_T has a naturally defined Fock vacuum “state” or functional and there is a corresponding inner-product space \mathcal{H}_T , $\langle \cdot, \cdot \rangle_T$, such that, in the associated GNS-representation, the identity $\langle a_i \varphi, \psi \rangle_T = \langle \varphi, a_i^* \psi \rangle_T$ holds. But the vacuum functional is generally not positive, and the operators in the representation not bounded, and therefore the Hermitian inner product $\langle \cdot, \cdot \rangle_T$ is then generally not positive semidefinite. The positivity question, and the faithfulness of the Fock representation, are the foci of this paper.

Note that (1) generalises some well-known types of deformed commutation relations, quantum groups, etc. (see [1, 3, 5, 6, 8, 12, 14, 15]). The basic examples for us will be the q_{ij} -CCR introduced and studied by M. Bożejko and R. Speicher (see [8, 12]), and the twisted canonical commutation relations (TCCR) constructed by W. Pusz and S.L. Woronowicz (see [6]). They were further studied in [16] where the traditional Cuntz algebra of [17] was considered as a base-point, corresponding to $q_{ij} = 0$, and the variation of the C^* -isomorphism class was considered as a function of q_{ij} .

EXAMPLE 1. q_{ij} -CCR, $2d$ generators:

$$\begin{aligned} \mathbb{C} \langle a_i, a_i^* \mid a_i^* a_j &= \delta_{ij} 1 + q_{ij} a_j a_i^*, \quad i, j = 1, \dots, d, \\ q_{ji} &= \bar{q}_{ij} \in \mathbb{C}, \quad |q_{ij}| \leq 1 \rangle \end{aligned}$$

EXAMPLE 2. The Wick algebra for TCCR:

$$\begin{aligned} a_i^* a_i &= 1 + \mu^2 a_i a_i^* - (1 - \mu^2) \sum_{k < i} a_k a_k^*, \quad i = 1, \dots, d \\ a_i^* a_j &= \mu a_j a_i^*, \quad i \neq j, \quad 0 < \mu < 1 \end{aligned}$$

We present some sufficient conditions on the coefficients $\{T_{ij}^{kl}\}$ for the existence of the Fock representation, and we describe the structure of the Fock space. We also give conditions for the faithfulness of Fock representation and describe its kernel in the degenerated case (see Sec. 3).

Further we consider the universal C^* -algebras for the examples above. Specifically we show that the universal C^* -algebras for q_{ij} -CCR (TCCR) can be generated by isometries (partial isometries) satisfying a certain algebraic relation. The description of the C^* -isomorphism classes for different values of parameters is presented.

We also show that the Fock representations of q_{ij} -CCR for some values of parameters, and TCCR for any value of parameter, are faithful on the C^* -level, i.e., the Fock representations of the corresponding C^* -algebras are faithful (see Sec. 4).

The complete proofs of all results presented here can be found in [4, 10, 11, 18, 19]. For detailed information about $*$ -representations of finitely generated $*$ -algebras see [20].

2. Basic definitions

Firstly let us construct a canonical realisation of Wick algebra, i.e., the $*$ -algebra on the relations (1), with coefficients $\{T_{ij}^{kl}\}$: we denote it by $W(T)$. To do it consider a finite-dimensional Hilbert space $\mathcal{H} = \langle e_1, \dots, e_d \rangle$. Construct the full tensor algebra over \mathcal{H} , \mathcal{H}^* , denoted by $\mathcal{T}(\mathcal{H}, \mathcal{H}^*)$. Then

$$W(T) \cong \mathcal{T}(\mathcal{H}, \mathcal{H}^*) / \langle e_i^* \otimes e_j - \delta_{ij} 1 - \sum T_{ij}^{kl} e_i \otimes e_j^* \rangle, \quad (2)$$

dividing out by the two-sided ideal on the relations (1). Note that in this realisation the subalgebra of $W(T)$ generated by $\{a_i\}$ is identified with the $\mathcal{T}(\mathcal{H})$.

The following operators were presented in [11] as a useful tool for computation with Wick algebras and their Fock representations.

$$\begin{aligned} T: \mathcal{H} \otimes \mathcal{H} &\mapsto \mathcal{H} \otimes \mathcal{H}, \quad T e_k \otimes e_l = \sum_{i,j} T_{ik}^{lj} e_i \otimes e_j, \quad T = T^* \\ T_i: \mathcal{H}^{\otimes n} &\mapsto \mathcal{H}^{\otimes n}, \quad T_i = \underbrace{1 \otimes \dots \otimes 1}_{i-1} \otimes T \otimes \underbrace{1 \otimes \dots \otimes 1}_{n-i-1}, \\ R_n: \mathcal{H}^{\otimes n} &\mapsto \mathcal{H}^{\otimes n}, \quad R_n = 1 + T_1 + T_1 T_2 + \dots + T_1 T_2 \dots T_{n-1}, \\ P_n: \mathcal{H}^{\otimes n} &\mapsto \mathcal{H}^{\otimes n}, \quad P_2 = R_2, \quad P_{n+1} = (1 \otimes P_n) R_{n+1}. \end{aligned} \quad (3)$$

The sequences of operators $P_0 = 1_{\text{vac}}$, $R_1 = 1 + T$, $P_1 = (1 \otimes 1)(1 + T) \cong 1 + T$, R_2, \dots, P_n are defined recursively. It is the sequence P_n which enters into the positivity question. The other one is only intermediate. The Hermitian inner product $\langle \cdot, \cdot \rangle_T$ on $\mathcal{T}_n(\mathcal{H})$ is then

$$\langle \phi, \psi \rangle_{\mathcal{T}_n(\mathcal{H})} := \langle \varphi, P_n \psi \rangle_{\text{tensor}} \quad (4)$$

where $\langle \cdot, \cdot \rangle_{\text{tensor}}$ is just the usual inner product on $\mathcal{T}_n(\mathcal{H})$ induced by $\langle \cdot, \cdot \rangle$ on \mathcal{H} . Hence, we need conditions on $T: \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ which make the operators P_n positive for all n . For example, in terms of these operators we can describe the procedure of Wick ordering, i.e., the commutation formula for fixed generator a_i^* and any homogeneous polynomial in a_k , $k = 1, \dots, d$ (see [21]).

Proposition 1. *Let $X \in \mathcal{H}^{\otimes n}$. Then*

$$e_i^* \otimes X = \mu(e_i^*) R_n X + \mu(e_i^*) \sum_{k=1}^d T_1 T_2 \cdots T_n (X \otimes e_k) e_k^*, \quad (5)$$

where $\mu(e_i^*): \mathcal{T}(\mathcal{H}) \mapsto \mathcal{T}(\mathcal{H})$ is defined as follows

$$\mu(e_i^*)1 = 0, \quad \mu(e_i^*)e_{i_1} \otimes \cdots \otimes e_{i_n} = \delta_{ii_1} e_{i_2} \otimes \cdots \otimes e_{i_n}.$$

For our examples the operator T have the following form:

EXAMPLE 3.

$$T e_i \otimes e_j = q_{ij} e_j \otimes e_i, i, j = 1, \dots, d.$$

EXAMPLE 4.

$$\begin{aligned} T e_i \otimes e_i &= \mu^2 e_i \otimes e_i \\ T e_i \otimes e_j &= \mu e_j \otimes e_i, \quad i < j \\ T e_i \otimes e_j &= -(1 - \mu^2) e_i \otimes e_j + \mu e_j \otimes e_i, \quad i > j. \end{aligned}$$

Note that for both examples, the operator T satisfies a **braid condition**, i.e., on the $\mathcal{H}^{\otimes 3}$ we have

$$T_1 T_2 T_1 = T_2 T_1 T_2. \quad (6)$$

The operators presented above appear naturally in construction of Fock representation of $W(T)$. This notion is induced in the obvious way from the classical one for CCR, however, in general, the Fock space is not always symmetric (see [11]).

Definition 1. The representation λ_0 acting on the space $\mathcal{T}(\mathcal{H})$ by formulas

$$\begin{aligned}\lambda_0(a_i)e_{i_1} \otimes \cdots \otimes e_{i_n} &= e_i \otimes e_{i_1} \otimes \cdots \otimes e_{i_n}, \quad n \in \mathbb{N} \cup \{0\} \\ \lambda_0(a_i^*)1_{\text{vac}} &= 0\end{aligned}$$

where the action of $\lambda_0(a_i^*)$ on the monomials of degree $n \geq 1$ is determined inductively using the basic relations, is called the Fock representation.

It is easy to see that $\lambda_0(a_i)$ are the classical creation operators and $\lambda_0(a_i^*)$ are twisted annihilation ones. Evidently in this way we have constructed a representation of $W(T)$, but not yet a $*$ -representation. To do it one has to supply the $\mathcal{T}(\mathcal{H})$ by the appropriate inner product (see [11]). This is where formula (4) comes in.

Definition 2. The Fock inner product (see [11]) is the unique semilinear Hermitian form $\langle \cdot, \cdot \rangle_T$ on $\mathcal{T}(\mathcal{H})$ such that

$$\langle \lambda_0(a_i)X, Y \rangle_T = \langle X, \lambda_0(a_i^*)Y \rangle_T, \quad X, Y \in \mathcal{T}(\mathcal{H}).$$

Similarly to the definition of Fock representation, the Fock inner product on $\mathcal{T}(\mathcal{H})$ can be computed inductively. It is easy to see that for $X \in \mathcal{H}^{\otimes m}$, $Y \in \mathcal{H}^{\otimes n}$, $n \neq m$, we have $\langle X, Y \rangle_T = 0$. On the components of powers 0, 1, the Fock inner product coincides with the standard one. For any $X, Y \in \mathcal{H}^{\otimes n}$, $n \geq 2$, we have

$$\langle X, Y \rangle_T = \langle X, P_n Y \rangle,$$

which agrees with (4) above. The operator $P_n = P_n(T)$ are given in (3).

Evidently, if we want to extend the Fock representation of $W(T)$ to the $*$ -representation by Hilbert-space operators, we should require that all the operators P_n , $n = 2, \dots$, be positive semidefinite, and that the subspace

$$\mathcal{I} = \bigoplus_{n \geq 2} \ker P_n$$

determines the kernel of the Fock inner product. Consequently the Hilbert-space structure of the Fock space emerges.

3. The structure of the Fock representation

In this section we present some sufficient conditions posed on the operator T for the positive-definite property of the Fock inner product, and we show that the kernel of the Fock representation is generated as a $*$ -ideal by the kernel of the Fock inner product. In particular, when the Fock inner product is strictly positive definite (i.e., when it has zero kernel), the Fock representation π_F is faithful, i.e., $\ker(\pi_F) = 0$.

There are several sufficient conditions on the operator T for the Fock inner product to be positive. It was shown in [10] that for sufficiently small coefficients we have strict positivity of the Fock inner product. This result is a corollary of the stability of the universal enveloping C^* -algebra for the Wick algebra around the zero base point (see Sec. 4).

Theorem 1. *If the operator T satisfies the norm bound $\|T\| < \sqrt{2} - 1$, then $P_n > 0$, $n \geq 2$, where $>$ refers to strict positivity.*

Another kind of sufficient condition is positivity of operator T (see [11]).

Theorem 2. *If $T \geq 0$ then $P_n > 0$, $n \geq 2$.*

In the present paper we will suppose that the operator T satisfies the **braid** condition (6). It was shown by M. Bożejko and R. Speicher (see [12]) that, in this case, the operators P_n , $n \geq 2$, have a natural description in terms of quasimultiplicative operator-valued mappings on the Coxeter group S_n . The following is a corollary of a much more general result proved in [12] for mappings on the general Coxeter group.

Theorem 3. *Let T satisfy the braid condition (6) and suppose $-1 \leq T \leq 1$. Then $P_n \geq 0$. Moreover, if $\|T\| \leq 1$, then $P_n > 0$, and the operators of the Fock representation are bounded, i.e., the Fock representation is by bounded operators. (Recall, the Fock representation of the undeformed CCR-algebra is unbounded.)*

We present a more precise version of this theorem. Namely, we give the description of kernel of P_n in the degenerate case. As an immediate corollary of this result we have the strict positivity of P_n , $n \geq 2$, for braided T satisfying the inequality $-1 < T \leq 1$ (see [4]).

Theorem 4. *Let $W(T)$ be a Wick algebra with braided operator T satisfying the norm bound $\|T\| \leq 1$. Then for any $n \geq 1$,*

$$\ker P_{n+1} = \sum_{k+l=n-1} \mathcal{H}^{\otimes k} \otimes \ker(1+T) \otimes \mathcal{H}^{\otimes l} = \sum_{k=1}^n \ker(1+T_k).$$

Let us illustrate this result on the examples.

EXAMPLE 5. For q_{ij} -CCR we have the alternatives:

– $|q_{ij}| < 1$ for any $i, j = 1, \dots, d$.

In this case $-1 < T < 1$ and the Fock inner product is strictly positive.

– $|q_{ij}| = 1$, $i \neq j$.

For these values of parameters we have $-1 \leq T \leq 1$ and

$$\ker(1+T) = \langle a_j a_i - q_{ij} a_i a_j, i < j \rangle.$$

EXAMPLE 6. For the TCCR Wick algebra, we have $-1 \leq T \leq 1$, and

$$\ker(1 + T) = \langle a_j a_i - \mu a_i a_j, i < j \rangle.$$

The following proposition shows that, for algebras with braided operator T , the kernel of the Fock representation is generated as a $*$ -ideal by the kernel of the Fock inner product, i.e.,

$$\mathcal{I} = \bigoplus_{n \geq 2} \ker P_n.$$

Proposition 2. *Let $W(T)$ be a Wick algebra with braided operator T and let the Fock representation λ_0 be positive (i.e., the Fock inner product is positive definite). Then*

$$\ker \lambda_0 = \mathcal{I} \otimes \mathcal{T}(\mathcal{H}^*) + \mathcal{T}(\mathcal{H}) \otimes \mathcal{I}^*.$$

Combining this proposition with Theorem 4, we get:

Theorem 5. *Let $W(T)$ be a Wick algebra with the braided operator T , $-1 \leq T \leq 1$. Then the kernel of the Fock representation is generated as a $*$ -ideal by $\ker(1 + T)$.*

This theorem implies that, for q_{ij} -CCR, $|q_{ij}| < 1$, the Fock representation is faithful. For the TCCR Wick algebra, and for q_{ij} -CCR, the kernels of the Fock representations are generated by the families $a_j a_i - \mu a_i a_j$, $i < j$, and $a_j a_i - q_{ij} a_i a_j$, $i < j$, respectively; and hence the Fock representations of quotients of these algebras by the $*$ -ideals generated by these families are faithful.

4. Universal bounded representation

In this section we discuss universal enveloping C^* -algebras for q_{ij} -CCR and Wick TCCR.

Let us recall that the universal C^* -algebra for a certain $*$ -algebra \mathcal{A} is also called the universal bounded representation. It is the C^* -algebra \mathbf{A} with natural homomorphism $\psi: \mathcal{A} \rightarrow \mathbf{A}$ such that, for any homomorphism $\varphi: \mathcal{A} \rightarrow B$, where B is a C^* -algebra, there exists a unique homomorphism $\theta: \mathbf{A} \rightarrow B$ satisfying $\theta \circ \psi = \varphi$. It can be obtained by the completion of \mathcal{A}/J with the following C^* -seminorm on \mathcal{A} :

$$\|a\| = \sup_{\pi} \|\pi(a)\|,$$

where \sup is taken over all bounded representations of \mathcal{A} , and J is the kernel of this seminorm. Obviously this process requires that $\sup_{\pi} \|\pi(a)\| < \infty$ for any $a \in \mathcal{A}$. Note that for our examples this condition is satisfied.

The universal bounded representation for q_{ij} -CCR was studied in [9, 10]. The following proposition follows from the main result of paper [10].

Proposition 3. *Let $\mathbf{A}_{\{q_{ij}\}}$ be the universal enveloping C^* -algebra for q_{ij} -CCR, $|q_{ij}| < \sqrt{2} - 1$. Then there exists the natural isomorphism*

$$\mathbf{A}_{\{q_{ij}\}} \cong \mathbf{A}_0,$$

where \mathbf{A}_0 is a C^* -algebra generated by the isometries s_i , $i = 1, \dots, d$, satisfying

$$s_i^* s_j = 0, \quad i \neq j$$

i.e., isomorphism with the Cuntz-Toeplitz algebra.

This implies that the Fock representation of $\mathbf{A}_{\{q_{ij}\}}$ is faithful.

Let us consider the $\mathbf{A}_{\{q_{ij}\}}$, $|q_{ij}| = 1$, for any $i \neq j$ and $q_{ii} := q_i$, $|q_i| < 1$ (i.e., unimodular off-diagonal terms). In this case, we do not have stability on the whole set of parameters (see [18]).

Proposition 4. *If for any $i \neq j$ we have $|q_{ij}| = 1$, then $\mathbf{A}_{\{q_{ij}\}}$ is isomorphic to the C^* -algebra $\mathbf{A}_{0, \{q_i\}}$ generated by isometries $\{s_i, i = 1, \dots, d\}$ satisfying*

$$s_i^* s_j = q_{ij} s_j s_i^*, \quad s_j s_i = q_{ij} s_i s_j, \quad i \neq j,$$

and the Fock representation of $\mathbf{A}_{\{q_{ij}\}}$ is faithful.

Finally for the universal C^* -algebra \mathbf{A}_μ for the Wick TCCR, we have the isomorphism $\mathbf{A}_\mu \cong \mathbf{A}_0$ for any $-1 < \mu < 1$, where the C^* -algebra \mathbf{A}_0 is generated by the partial isometries $\{s_i, i = 1, \dots, d\}$ satisfying the relations

$$s_i^* s_j = \delta_{ij} \left(1 - \sum_{k < i} s_k s_k^* \right), \quad i, j = 1, \dots, d.$$

The Fock representation of \mathbf{A}_μ is faithful also (see [19]).

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